Answer Key

Test # 1

Free Response Questions

Section II

SECTION II, PART A
Time—30 minutes
Number of problems—2

A graphing calculator is required for some problems or parts of problems.

During the timed portion for Part A, you may work only on the problems in Part A.

On Part A, you are permitted to use your calculator to solve an equation, find the derivative of a function at a point, or calculate the value of a definite integral. However, you must clearly indicate the setup of your problem, namely the equation, function, or integral you are using. If you use other built-in features or programs, you must show the mathematical steps necessary to produce your results.

1. An object is moving along the x-axis with its velocity given by $v(t)=\dfrac{2\sin(1.2t^3)}{1+t^2}$, where t is time measured in seconds and $0\le t\le 5$. The object's initial position is at $x=8$.

   (a) Find the acceleration of the object at the time $t=4$.

   (b) Find the position of the object at $t=4$.

   (c) What is the distance that the object travels in the interval $0\le t\le 5$?

   (d) What is the displacement of the object in the interval $0\le t\le 5$?

   (e) If a different object moves along the x-axis with its position given by $x_2=t^3-t^2$, at what time t are the two objects traveling with the same velocity?

a) $a(t)=v'(t)=\dfrac{(1+t^2)(2\cos(1.2t^3))(3.6t^2)-(2\sin(1.2t^3))(2t)}{(1+t^2)^2}$

$a(4)=1.085$

b) $s(t)=8+\int_0^4 v(t)\,dt$

$=8+\int_0^4\left(\dfrac{2\sin(1.2t^3)}{1+t^2}\right)dt$

$=8+0.494 \implies$ position is 8.494

GO ON TO THE NEXT PAGE.

c) distance $= \int_0^5 |v(t)|\,dt$

$$= \int_0^5 \left|\left(\frac{2\sin(1.2t^3)}{1+t^2}\right)\right| dt$$

$= 1.130$.

d). displacement $= \int_0^5 v(t)\,dt$

$$= \int_0^5 \left(\frac{2\sin(1.2t^3)}{1+t^2}\right) dt$$

$= 0.494$

e) $x_2 = t^3 - t^2$

$x_2' = 3t^2 - 2t$

Set both velocities equal to each other & solve.

$$3t^2 - 2t = \frac{2\sin(1.2t^3)}{1+t^2}$$

Graph on calc to find intersection

$t = 0.980$, $t = 0$

when velocity is 0, then both particles are stopped.

At $t = 0.980$ secs both particles are travelling at the same velocity.

Section II

2. People board a train at a rate modeled by the function B given by

$$B(t)=\begin{cases}1800\left(\frac{t}{10}\right)^2\left(1-\frac{t}{10}\right)^3; & 0 \le t \le 10 \\ 0; & t>10\end{cases}$$ where $B(t)$ is measured in people per minute and t is measured in minutes.

where $B(t)$ is measured in people per minute and t is measured in minutes. As people board the train, they exit at a constant rate rate of 3.2 people per minute. There are initially 30 people on the train. $(0, 30)$

(a) How many people board the train during the time interval $0 \le t \le 10$?

(b) How many people are on the train at time $t = 10$?

(c) No one boards the train after time $t = 10$, so at what time will there be no people on the train?

(d) At what time t is the number of people on the train a maximum?

a) # of people boarding for $0 \le t \le 10$ $= \int_0^{10} B(t)\,dt$

$= 300$ people

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b) # of people on train at $t=10$

= initial amt + # of people boarding − # of people exiting

$= 30 + \int_0^{10} (B(t) - 3.2)\,dt$

$= 30 + 268$

$= 298$ people.

c). to find time, # of people at $t=10$ is 298.

So $t = 10 + \dfrac{298 \text{ people}}{\frac{3.2 \text{ people}}{1 \text{ min}}}$

$= 10 + 93.125$ mins

At 103.125 mins there will be no one left on the train.

20.

d). At ? t is # of people on board a max.

Total # on board found at (b)

$= 30 + \int_0^t (B(t) - 3.2)\,dt \Rightarrow$ at $t = 10$, 298 people.

Set $\frac{d}{dt}\int_0^t (B(t) - 3.2) = 0$ and solve for c.p's

Graph $B(t) - 3.2$ and find zeros.

at $t = 0.451$ and $t = 8.667$.

Make a table showing c.p's & endpoints for abs max.

t	Total people on board (see eq. above)
0	30 → Given.
0.451	29.053
8.667	301.131
10	298 → calculated at (b).

Max # of people is 301 people at $t = 8.667$ mins

SECTION II, PART B
Time—1 hour
Number of problems—4

No calculator is allowed for these problems.

During the timed portion for Part B, you may continue to work on the problems in Part A without the use of any calculator.

$r = 2$

3. Grain is filling a container that is in the shape of a $\boxed{\text{right circular cylinder}}$ with a $\boxed{\text{diameter of 4 feet.}}$ The $\boxed{\text{rate of change of the height of the grain in the cylinder is given by } \frac{dh}{dt} = 12\sqrt{4+h},}$ where h is measured in feet and t is time measured in minutes.

(a) Find the rate of change of the volume of grain in the container with respect to time when $\boxed{h = 12 \text{ feet.}}$

(b) When the height of the grain is 12 feet, how fast is the rate of change of the height decreasing?

(c) $\boxed{\text{Initially, the container of grain is empty.}}$ Find an expression for h in terms of t.

I.C $= (0,0)$

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a) $V(\text{cyl}) = \pi r^2 h \Rightarrow V = 4\pi h$

Need $\frac{dV}{dt}$? Given $\frac{dh}{dt} = 12\sqrt{4+h}$

$$\frac{dV}{dt} = 4\pi \frac{dh}{dt}$$

$$\frac{dV}{dt} = 4\pi\left(12\sqrt{4+h}\right)$$

$$\left.\frac{dV}{dt}\right|_{h=12} = 4\pi\left(12\sqrt{16}\right) = 4\pi(48)$$

$$\boxed{\frac{dV}{dt} = 192 \text{ ft}^3/\text{min}}$$

b) Need $\frac{d^2h}{dt^2}$ $\Rightarrow$

$$\frac{dh}{dt} = 12(4+h)^{1/2}$$

$$\frac{d^2h}{dt^2} = 12\left(\frac{1}{2}\right)(4+h)^{-1/2} \cdot 1 \frac{dh}{dt}$$

$$= \frac{6}{\sqrt{4+h}} \frac{dh}{dt}$$

$$\left.\frac{d^2h}{dt^2}\right|_{h=12} = \frac{6}{\sqrt{4+12}} \cdot \left(12\sqrt{4+12}\right)$$

$$= \frac{6}{\cancel{4}}(\overset{12}{\cancel{48}})$$

$$= 72 \text{ ft/min}^2$$

c) Need to solve for h.

$$\frac{dh}{dt} = 12\sqrt{4+h}$$

$$\frac{dh}{\sqrt{4+h}} = 12\, dt$$

$$\int (4+h)^{-1/2} dh \quad 12\int dt$$

$$\frac{4+h^{1/2}}{\frac{1}{2}} = 12t + C$$

$$2\sqrt{4+h} = 12t + C$$ Initial condition = (0,0)

$$2\sqrt{4+0} = 12(0) + C$$

$$4 = C$$

$$2\sqrt{4+h} = 12t + 4$$ Solve for h.

$$\left(\sqrt{4+h}\right)^2 = (6t+2)^2$$

$$4+h = (6t+2)^2$$

$$\boxed{h = (6t+2)^2 - 4}$$

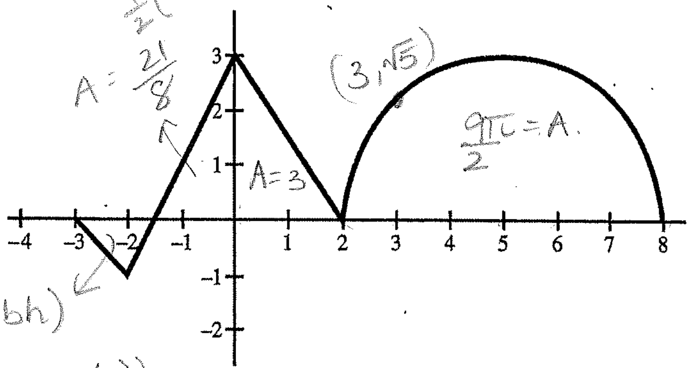
Section II

4.

$\frac{1}{2}bh$

$\frac{1}{2}(1.75(3))$

$A = \frac{21}{8}$



$A = -(\frac{1}{2}bh)$

$A = -\frac{1}{2}(1.25(1))$

$A = -\frac{5}{8}$

The continuous function f is defined on the interval $[-3, 10]$. The graph on the interval $[2, 8]$ is a semicircle. The figure above shows a portion of the graph of f. The graph contains the point $(3, \sqrt{5})$ and crosses the x-axis at the point $\left(-\frac{7}{4}, 0\right)$.

(a) If $\int_0^{10} f(x)\,dx = 20$, what is the value of $\int_8^{10} f(x)\,dx$?

(b) Evaluate $\int_2^3 4f'(x)\,dx$.

(c) If $g(x) = \int_{-3}^{x} f(x)\,dx$, find the absolute maximum of g on the interval $[-2, 8]$. Justify your answer.

(d) Evaluate $\lim_{x \to -1} \frac{3f(x) - 3}{2(x+1)}$.

a) $\int_0^8 f(x)dx + \int_8^{10} f(x)dx = \int_0^{10} f(x)dx$

$3 + \frac{9}{2}\pi + \quad = 20$

So $\int_8^{10} f(x)dx = 20 - (3 + \frac{9}{2}\pi)$

$= 17 - \frac{9}{2}\pi$

$g(x) = \int_{-3}^{x} f(x)dx$

$g'(x) = \frac{d}{dx}\int_{-3}^{x} f(x)dx$ FTC I

$= f(x)dx$

b) $\int_2^3 4f'(x)dx = 4\int_2^3 f'(x)dx$ → FTC I

$= 4[f(3) - f(2)]$

$= 4(\sqrt{5} - 0)$

$= 4\sqrt{5}$

22.

c) $g(x) = \int_{-3}^{x} f(x)dx$

where $f(x)=0$

$g'(x) = f(x)$ To find abs max, we need c.p's and endpoints

x	$g(x)$
-3	0
$-\frac{7}{4}$	$-\frac{5}{8}$
2	5
8	$5+\frac{9}{2}\pi$

$\int_{-3}^{-3} f(x)dx$

$\int_{-3}^{-7/4} f(x)dx$

$\int_{-3}^{2} f(x)dx \to \left(-\frac{5}{8}+\frac{21}{8}+3\right)$

Abs max occurs at $[-2,8]$ and is $5+\frac{9}{2}\pi$

d) $\lim\limits_{x\to -1} \frac{3f(x)-3}{2(x+1)}$ $\quad \frac{3(1)-3}{2(0)}$ $\left(\frac{0}{0}\right)$

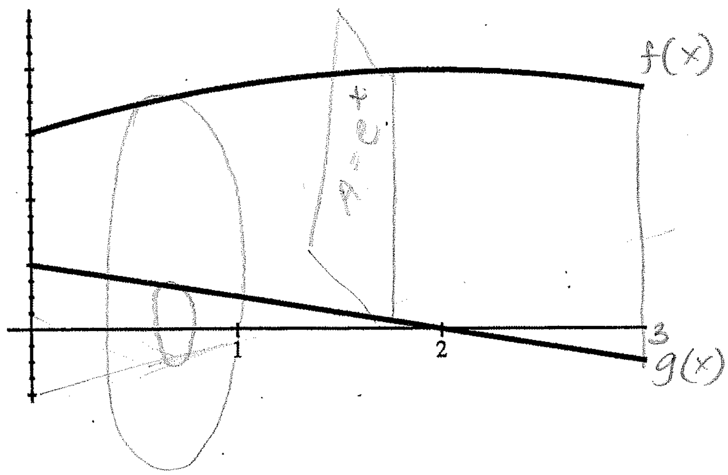
$\lim\limits_{x\to -1} \frac{3f'(x)}{2} = \frac{3(2)}{2} = 3.$

$\to$ slope @ $x=-1$ $\frac{y_2-y_1}{x_2-x_1}$

$\frac{3-(-1)}{0-(-2)} = \frac{4}{2}$

$= 2$

5.



Let R be the region enclosed by the graphs of $f(x) = 16 - (x-2)^2$, $g(x) = 4 - 2x$, the y-axis and the line $x = 3$, as shown above.

(a) Find the area of R.

(b) If the region R is the base of a solid and at each x, the perpendicular cross-section to the x-axis has an area $A(x) = e^x$, find the volume of the solid.

(c) *Set up but do not evaluate* the integral that gives the volume of the solid that is generated when R is rotated about the x-axis.

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a) Area $= \int_0^3 \left[\left(16-(x-2)^2\right) - (4-2x)\right] dx$

$= \int_0^3 \left[16-(x^2-4x+4) - 4 + 2x\right] dx$

$= \int_0^3 \left[16 - x^2 + 4x - 4 - 4 + 2x\right] dx$

$= \int_0^3 (8 - x^2 + 6x)\, dx$

$= 8x - \frac{1}{3}x^3 + 3x^2 \Big|_0^3$

$= 8(3) - \frac{3^3}{3} + 3(3)^2 - 0$

$24 - 9 + 27$

$= 42$

b) $\text{Vol} = \int_0^3 A(x)\,dx$

$= \int_0^3 e^x\,dx$

$= e^x \Big|_0^3 = e^3 - e^0$

$\boxed{V = e^3 - 1}$

c) Washers:

$\text{Vol} = \pi \int_0^3 \left[\left(16 - (x-2)^2\right) - (4-2x)^2\right] dx$

Section II

6. Functions f, g, and h are differentiable functions and $f(4) = g(4) = 10$. The line $y = 6 - 3(x - 1)$ is tangent the graphs of f and g at $x = 4$.

   (a) Find $f'(4)$.

   (b) Let w be the function given by $w(x) = \frac{x^4}{64} g(x)$.

   (i) Find an expression for $w'(x)$.

   (ii) Evaluate $w'(4)$.

   (c) If the function h is defined by $h(x) = \frac{16 - x^2}{100 - g(x)^2}$ for $x \neq 4$, find $\lim_{x \to 4} h(x)$.

$y = -3x + 3 + 6$

$y = -3x + 9$

$\downarrow$

m

$y = -3(4) + 9$

$= -12 + 9$

$= -3$

a) $f'(4) = \text{slope of tangent at } x = 4$

$= -3$

b) i) $w(x) = \frac{x^4}{64} g(x)$

$w'(x) = \frac{1}{64}\left(x^4 g'(x) + g(x)\, 4x^3\right)$

ii) $w'(4) = \frac{1}{64}\left(4^4 g'(4) + g(4)\, 4(4)^3\right)$

$= \frac{1}{64}\left(256(-3) + 10(256)\right)$

$= \frac{1}{64}\left(-768 + 2560\right)$

$= 28$

c) $\lim_{h \to 4} \frac{16 - x^2}{100 - g(x)^2} \quad \left(\frac{0}{0}\right)$

$\lim_{h \to 4} \frac{-2x}{-2g(x) \cdot g'(x)}$

$\lim_{h \to 4} \frac{x}{g(x) \cdot g'(x)}$

$= \frac{4}{10(-3)} = \frac{4}{-30} = -\frac{2}{15}$

STOP

END OF EXAM

24.