CHAPTER 5: THE DEFINITE INTEGRAL

Section 5-1: Estimating with Finite Sums

Objectives:

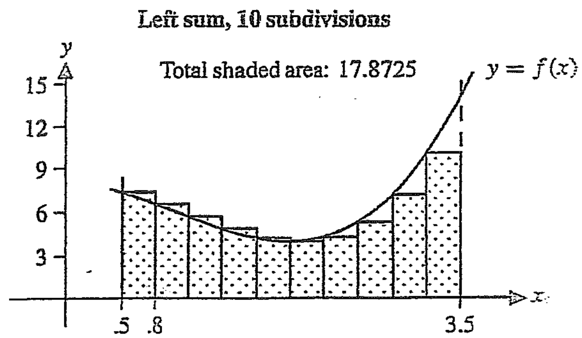
- **Approximate the area under the graph of a nonnegative, continuous function by using rectangle approximations.**
- **Interpret the area under a graph as a net accumulation of a rate of change.**

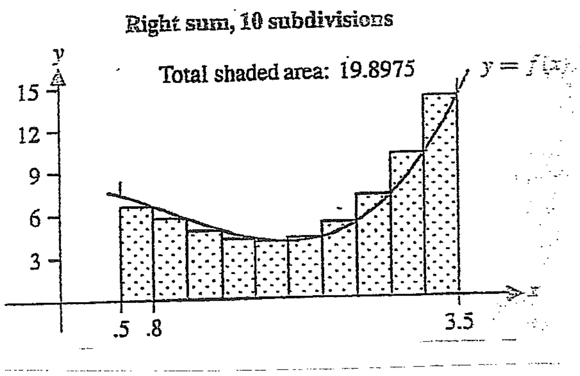
1. **Two Ideas – vast implications:** The need to calculate instantaneous rates of change led to an investigation of the slopes of tangent lines and ultimately, to the ______________________________. Derivatives reveal only half the story. In addition to calculating and describing how functions change at any given instant, early mathematicians needed a method to describe how instantaneous changes could accumulate over an interval to produce the function. This need led them to investigate the areas under curves, which is the second main branch of calculus, called ________________________.

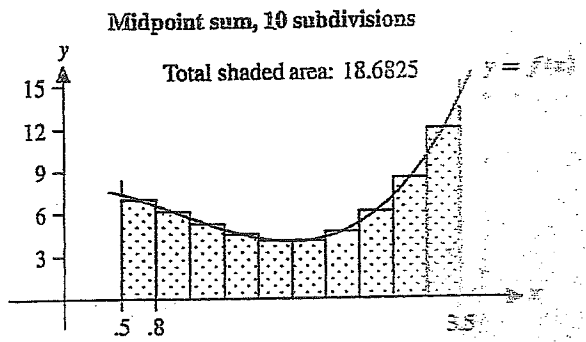
Two geometric views of the above paragraph:

2. **A simple car illustration:**

3. **More illustrations to help with the concept of integral:**







4. **Important ideas:**

a) The illustrations above demonstrate the **rectangular approximation method (RAM)** to approximate the area under the curve. The variances in the names suggest the choice we made:
LRAM – left-hand endpoint
RRAM – right-hand endpoint
MRAM – midpoint endpoint

b) Remember the sandwich theorem? When we use the rectangular approximation method, we use an underestimate and an overestimate. The exact answer lies somewhere between these two estimates. How can we get a better approximation? We make the sandwich thinner by taking smaller intervals of time.

c) No one computation will give us the exact answer. Instead the exact answer is a process of doing something infinitely many times. In the car example, the process involved partitioning the time into smaller and smaller increments, calculating the area of each by multiplying each $\Delta t \times v(\Delta t)$ and then summing them together. This summation gets closer to an exact answer as Δt gets smaller. And in the **limit** of this summation we get the single answer.

d) The concepts of calculus that are at the heart of the derivative and the integral is this idea of the limiting process.

e) Rectangles are not the only figure used to approximate areas under curves. We will also study a trapezoidal approximation method in another section.

f) Use of the integral helps us solve problems concerning volumes, lengths of curves, population predictions, cardiac output, work related problems, consumer surplus and many others.

Example 1: Using 10 rectangles, estimate the area under $f(x) = \ln(x)$ from $x = 1$ *to 6.* Use RRAM, LRAM and MRAM. Provide a sketch for each method and show all the work.

Assignment: Pg 270 #'s 2-4, 9, 16, 20a, 24, 28, 30

Section 5-1/2: Review and Definite Integrals

Objectives:

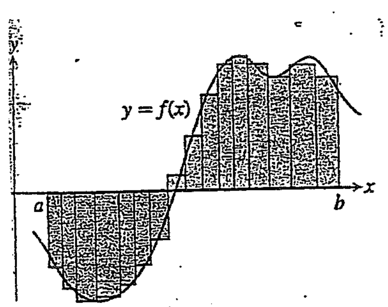
- **Express the area under a curve as a definite integral and as a limit of Riemann sums.**
- **Compute the area under a curve using a numerical integration procedure.**

A. Overview

a) In Geometry, there was a way to approximate the area of an irregular shape like a lake or state.
b) Integral Calculus is used to find lengths, areas and volumes of figures that have no standard formulas to use.
c) Also, we can predict future population sizes and future costs of living among other things.

B. Regions bounded by curves

1. DesCartes and Fermat approximated the area of a region under a curve by using rectangles whose "height" is given by the curve itself.

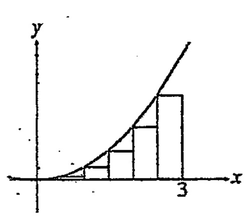


← if we add up the areas of all the rectangles here, we should be pretty close to the area under the curve between a and b.

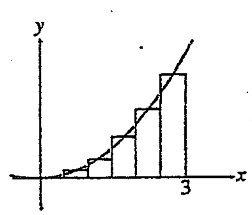
2. As the rectangles get thinner, our approximation improves.
3. All we need is:
   a) A way to write formulas for sums of many terms.
   b) A way to find numerical limits as the number of terms increases without bound.

C. Areas under a Curve

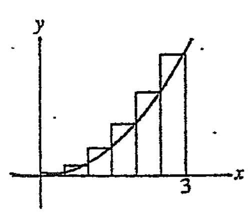
1. **Notation:** $A_a^b f(x)$ = Area under the curve of f(x) between a and b.
   ex. $A_0^5 x^2$ = the area under $y = x^2$ between 0 and 5.
2. We need to "partition" off rectangles.
   $x_0, x_1, x_2, \ldots, x_n$ are the "starting" points of each rectangle.
3. RAM



LRAM



MRAM



RRAM

- Area of each rect = ht x width
  $= f(x_i) \bullet \Delta x_i$

4. **Definition:** $A_a^b f(x) = \lim_{n\to\infty} LRAM_n f(x) = \lim_{n\to\infty} RRAM_n f(x) = \lim_{n\to\infty} MRAM_n f(x)$

In English, "Area under the curve $y = f(x)$ from a to b."

D. Riemann Sums

1. Sigma Notation Review

$$a_1 + a_2 + ... + a_n = \sum_{k=1}^{n} a_k$$ index = k, $a_k = k^{th}$ term

2. A Riemann sum is the sum of the areas of all the rectangles under a curve

$$S_p = \sum_{k=1}^{n} f(x_k)\Delta x_k$$ S_p is the Sum of the Partitions

3. LRAM, MRAM, and RRAM are all examples of Riemann Sums

4. This sum depends on the number of partitions chosen, and the point in the subinterval, x_k, that we choose (left endpoint, right endpoint, or midpoint).

E. Integrals

1. **Definition:** $\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k = I$, where I is called the <u>definite integral</u> of f over [a,b].

We say that f is integrable over [a,b] and:

$$I = \int_a^b f(x)dx$$ ← "the integral of f from a to b"

$\int_a^b f(x)dx$

Integration terms:

$\int$ = integral sign

to integrate = evaluate the integral

upper limit of integration = b

lower limit of integration = a

x = variable of integration

F. Putting it all together

$$Area = A_a^b f(x) = \int_a^b f(x)dx$$

1. For any integrable function, $\int_a^b f(x)dx =$ (area above x-axis) - (area below x-axis).

2. Constant Function: If $f(x) = c$, then $\int_a^b f(x)dx = \int_a^b c\,dx = c(b-a)$

c = height, (b-a) = width

3. Your calculator can approximate the area under the curve.

$NINT(f(x),x,a,b)$ finds the area under the curve of *f(x)* between *a* and *b*.
NINT (function, variable of integration, lower limit of integration, upper limit of integration)

Example 1: $NINT(x^2,x,0,5)$

Example 2: $\int_{-1}^{2} x\sin x\,dx$

Example 3: Evaluate $\int_{3}^{7}(-20)dx$

Example 4: Use the graph of the integrand and area formulas to evaluate:

$\int_{-1}^{1}(1-|x|)dx$

4. Another view: Greek vs Roman

Assignment: Pg 282 #'s 1-39 odds

Section 5-3: Definite Integrals and Antiderivatives

Objectives:

- **Apply rules for definite integrals and find the average value o a function over a closed interval.**

A. **Properties of Definite Integrals** (See Pg 285)

1. **Order of Integration:** When we defined the definite integral $\int_a^b f(x)dx$ we implicitly assumed that a<b. But the definition as a limit of Riemann sums makes sense even if a>b. Notice that this changes Δx from $(b-a)/n$ to $(a-b)/n$. Therefore

2. **Zero:**

3. **Constant Multiple:**

4. **Sum and Difference:**

5. **Additivity:**

6. **Max-Min Inequality:**

7. **Domination:**

B. **Average value of a function**

1. **Definition:** <u>The average value of f on [a,b]</u> is $av(f) = \frac{1}{b-a}\int_a^b f(x)dx$.

   If f is non-negative, this suggests the average "height" of the graph.

2. **Example 1:** a) Use *fnInt* to find the average value of the function $f(x) = 4 - x^2$ on the interval [0,3].
   b) At what point(s) in the interval does the function assume its average value?

C. **Mean Value Theorem for Definite Integrals**

1. **Definition:** If f is continuous on [*a, b*], then there exists a number c in [*a, b*] such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

2. The geometric interpretation:

You can always chop off the top of a two-dimensional mountain at a certain height and use it to fill in the valleys so that the mountain becomes completely flat!

Example 2: Find the average value of $f(x)=1+x^2$ on the interval [-1, 2]. Find the value(s) of c that confirm the mean value theorem for definite integrals.

D. **Differential Calculus meets Integral Calculus**

1. Read the section on Pg 288.
2. In a few sentences write the gist of these two paragraphs below.

E. **How to find the area under a curve from *a* to *b*:**

1. Find an antiderivative *(F(x))*
2. Calculate *F(b) – F(a).* This number will be $A_a^b f(x)$.

So: $$\int_a^b f(x)dx = F(b)-F(a)$$

This is part of the Fundamental Theorem of Calculus that we will look at tomorrow.

Assignment: Pg 291 3's 1-6, 7, 8, 11-35 odds

Section 5-4: Fundamental Theorem of Calculus

Objectives:

- **Apply the Fundamental Theorem of Calculus**
- **Understand the relationship between the derivative and definite integral as expressed in both parts of the Fundamental Theorem of Calculus.**

Part 1:

The Fundamental Theorem of Calculus (FTC) is appropriately named because it establishes a connection between the two branches of calculus: differential calculus and integral calculus. Differential calculus arose from the **tangent** problem, whereas integral calculus arose from a seemingly unrelated problem, the **area** problem. Newton's teacher at Cambridge, Isaac Barrow (1630-1677), discovered that these two problems are actually closely related. In fact, he realized that differentiation and integration are inverse processes. The FTC gives the precise inverse relationship between the derivative and the integral. It was Newton and Leibniz who exploited this relationship and used it to develop calculus into a systematic mathematical method. In particular, they saw that the FTC enabled them to compute areas and integrals very easily without having to compute them as limits of sums as we did in our previous sections in this chapter.

The FTC consists of two parts. The first part deals with functions defined by an equation of the form

$$g(x) = \int_a^x f(t)dt$$

where f is a continuous function on $[a, b]$ and x varies between a and b. Notice that g depends only on x, which appears as the variable upper limit in the integral. If x is a fixed number, then the integral $\int_a^x f(t)dt$ is a definite number. If f happens to be a positive function, then $g(x)$ can be interpreted as the area under the graph of f from a to x, where x can vary from a to b. Think of g as the "area so far" function. (See Figure 1).

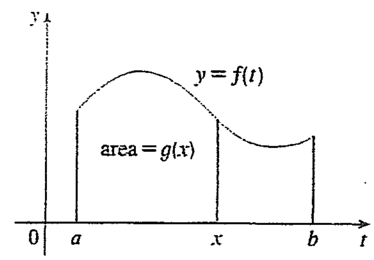


FIGURE 1

An example to illustrate Part 1 of the FTC:

If f is the function whose graph is shown in Figure 2 and $g(x) = \int_a^x f(t)dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$. Graph g.
Note: The required values are calculated for you ☺ Use the calculated values to sketch the graph of g. What do you notice about the graph of g in relationship to the graph of f in Figure 2? Write a statement below.

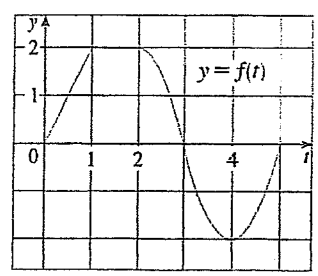


FIGURE 2

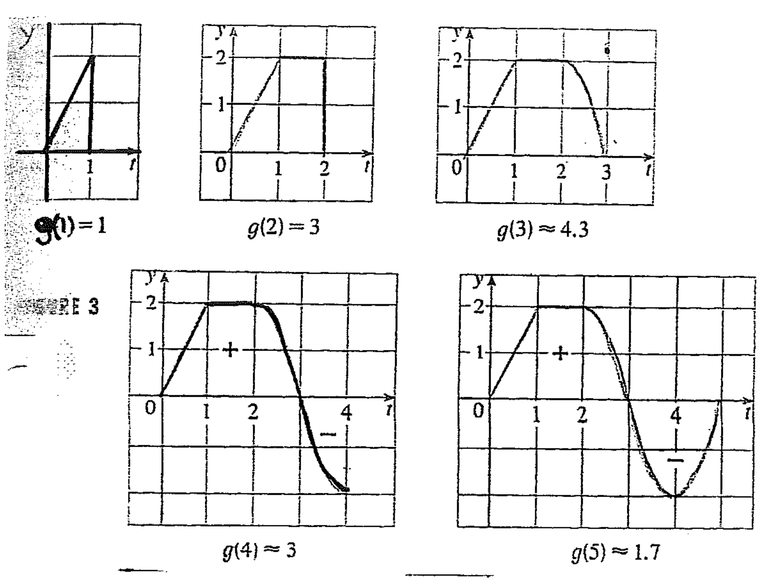


FIGURE 3

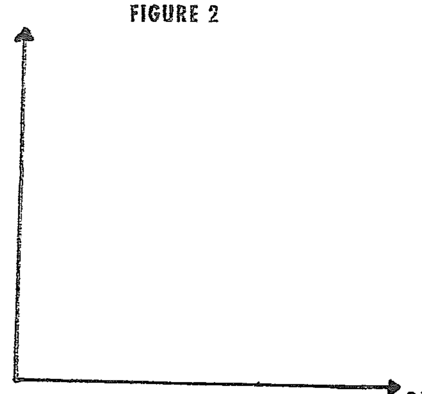
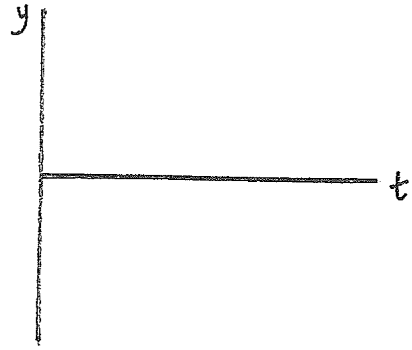


Fig 4

In other words, if *g* is defined as the integral of *f*, $g(x) = \int_a^x f(t)dt$, then *g* turns out to be the antiderivative of *f*. Sketch the derivative of *g* and see what you get.



To see why this might be generally true we consider any continuous function *f* with *f(x)* ≥ 0. Then $g(x) = \int_a^x f(t)dt$ can be interpreted as the area under the graph of *f* from *a* to *x*, as in Figure 1.

In order to compute $g'(x)$ from the derivative we first observe that, for *h* > 0, *g(x + h)* – *g(x)* is obtained by subtracting areas, so it is the area under the graph of *f* from *x* to *x* + *h* (Figure 5). For small *h* you can see from the figure that this area is approximately equal to the area of the rectangle with height *f(x)* and width *h*:

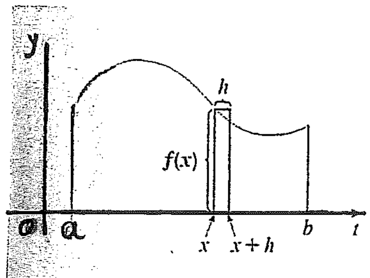


FIGURE 5

$$g(x+h) - g(x) \approx hf(x)$$

So

$$\frac{g(x+h) - g(x)}{h} \approx f(x)$$

Intuitively, we therefore expect that $g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x)$

The fact that this is true, even when f is not necessarily positive, is the first part of the FTC.

Fundamental Theorem Part I

If *f* is continuous on [*a*,*b*], then the function $F(x) = \int_a^x f(t)dt$ has a derivative at every point in [*a*,*b*] and

$$\frac{dF}{dt} = \frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Roughly speaking: The above equation says that if we first integrate *f* and then differentiate the result, we get back to the original function *f* evaluated at the **upper limit.**

Examples of the above theorem:

a) $\frac{d}{dx}\int_{-\pi}^{x} \cos t dt =$

b) $\frac{d}{dx}\int_a^x \frac{1}{1+t^2} dt =$

c) What if the upper limit of integration isn't just *x*? Find $\frac{d}{dx}\int_{1}^{x^4} \sec(t)dt$.

In general,

d) What if the variable is the lower limit of integration? Find $\frac{d}{dx}\int_{x^2}^{1} \cos t dt =$

In general,

Part 2:

Early in this chapter we computed integrals from the definition as a limit of Riemann sums and we saw that this procedure is sometimes long and difficult. The second part of the FTC, which follows easily from the first part, provides us with a much simpler method for the evaluation of integrals.

2. **Fundamental Theorem Part II**

If *f* is continuous on [a,b], then and *F* is any antiderivative of *f* on [a,b], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where *F* is any antiderivative of *f* , that is, a function such that *F'* = *f.*

Simply stated: If we know an antiderivative *F* of *f,* then we can evaluate $\int_a^b f(x)dx$ simply by subtracting the values of *F* at the endpoints of the interval [*a, b*]. It's very surprising that $\int_a^b f(x)dx$ which was defined by a complicated procedure can be found by knowing the values of *F(x)* at only two points, *a* and *b*.

The FTC Part II, finds the ________________ area (could be negative) from *a* to *b*.

On Calculator:

What if we wanted to find the ***area of a region*** bounded by a curve, or the ***total area*?**
Read the paragraph "Area Connection" on Pg 300. Very important!!

Example: Find the area of the region (or total area) between *f(x) = sin x dx*, $0 \le x \le 3\pi$, and the x-axis.

On Calculator:

Assignment: Pg 302 #'s 1-19 odds, 27-47 odds, 57, 59

Section 5-5: Trapezoidal Rule

Objectives:

- **Approximate the definite integral by using the Trapezoidal Rule.**

Review: What are LRAM, RRAM and MRAM? ____

____

1. The trapezoidal rule uses trapezoids to ______________________________________

____

Sketch and formula:

2. Why use it?
   1st – Trapezoids fit more accurately than ______________________________________
   2nd – Not all functions can be easily ____
   3rd – Most importantly, ____
3. Concavity: If a function is concave up on $[a,b]$ then T_n ____________________the integral
   and if a function is concave down then T_n _______________________ the integral.
4. Derivation of the trapezoidal method:

4. **Example 1**: Estimate the area under the curve of $f(x) = \ln(x)$ from x= 1 to 6 using 10 trapezoids.
   Provide a sketch

Assignment: Pg 312 #'s 2-8 evens, 9-12

Chapter 5 Test Review

The test will take one day and you may use your calculator, but show all algebraic thinking for full credit. The following topics will be covered on the test.

- Estimating with finite (Riemann) sums. (LRAM, RRAM, MRAM, Trapezoidal)
- Working with sigma notation
- Evaluating integrals by anti-differentiation
- Evaluating integrals with your calculator
- Expressing problems using the definite integral
- Properties of definite integrals
- Average (mean) value of a function: av(f(x))
- Mean Value Theorem for definite integrals
- Finding total and net area under a curve
- Analyzing anti-derivatives graphically
- Understanding and applying the Fundamental Theorem of Calculus (both parts)
- Writing about any of the above concepts, especially the FTC I and II, area connections and the concept of the integral (Riemann Sums).

Review Assignment: Pg 270 #'s 18, 29; Pg 282 #'s 4, 10, 16, 32; Pg 291 #'s 14, 22, 24, 30, 32;
Pg 302 #'s 10, 18, 30, 38, 46