CHAPTER 7 – APPLICATIONS OF DEFINITE INTEGRALS

Section 7-1: Integral as Net Change (Part 1)

Objective:
  • **Solve problems in which a rate is integrated to find the net change over time in a variety of applications.**

1. Part 2 of the FTC says that if f is continuous on [a, b}, then $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f. This means that $F' = f$, so the equation can be rewritten as

$\int_a^b F'(x)dx = F(b) - F(a)$.

We know that $F'(x)$ represents the _____ of $y = F(x)$ with respect to x and $F(b) - F(a)$ represents the _____ when x changes from _____.
Although y might change in both directions $F(b) - F(a)$ represents the _____ change in y. So we can reformulate the FTC2 in words as follows:

The Net Change Theorem: The integral of a rate of change is the net change:

$$\int_a^b F'(x)dx = F(b) - F(a).$$

2. This principle can be applied to all rates of change in the natural and social science examples.

3. **A Particle Example:** If an object moves along a straight line with position function $s(t)$, then its velocity $is\ v(t)\ = s'(t)$, so $\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$ is the net change of position, or **displacement,** of the particle during the time period from t_1 to t_2.

4. If we want to calculate the distance traveled during the time interval, we have to consider the intervals when $v(t) \geq 0$ (the particle moves to the right) and also the intervals when $v(t) \leq 0$ (the particle moves to the left). In both cases the distance is computed by integrating $|v(t)|$, the speed.
Therefore $\int_{t_1}^{t_2} |v(t)|dt$ = total distance traveled.

5. A sketch to help understanding:

6. The acceleration of the object is *a(t) = v′(t)*, so $\int_{t_1}^{t_2} a(t)dt = v(t_2) - v(t_1)$ is the change in velocity from t_1 to t_2.

7. Note: To integrate the absolute value of *v(t)*, we split the integral into two parts, one where *v(t)* ≤ *0* and one where *v(t)* ≥ *0*.

8. **Another way of saying the same thing:** Relationship between position, velocity, acceleration:

   - $\dfrac{ds}{dt} = s' = v(t)$ → Derivative of position is velocity

   - $\dfrac{dv}{dt} = v' = a(t)$ → Derivative of velocity is acceleration

9. **Example 1:** A particle moves along a line so that its velocity at time *t* is $v(t) = t^2 - t - 6$ (measured in meters per second).
a) Find the displacement of the particle during the time period *1 ≤ t ≤ 4*.
b) Find the distance traveled during this time period.

10. **Example 2:** The function $v(t)$ is the velocity in m/sec of a particle moving along the *x*-axis. Use analytic methods **(NO CALCULATOR)** to

a) Determine when the particle is moving to the right, to the left and stopped.

b) Find the particle's displacement for the given time interval. If $s(0) = 3$, what is the particle's final position?

$$v(t) = 6\sin(3t); \qquad 0 \le t \le \frac{\pi}{2}$$

B. Key Concepts/Conclusions

1. Integrating Velocity finds Displacement (difference between initial and final position)
   -Can also be found by calculating the <u>net</u> area b/w the velocity curve and time axis

2. Integrating the <u>absolute value</u> of Velocity gives the Total Distance Traveled.
   -Can also be found by calculating the <u>total</u> area between velocity curve and time axis

3. Whenever you want to find the cumulative effect of a varying rate of change, integrate it.
   - Integrals can be used to calculate growth, decay, and consumption (see ex. p383-385)

Assignment: **R**ead Pg 378-383. Day 1: Pg 386 #'s 1-11 odd, 12-17, 20. Use analytical methods as much as possible.

Section 7-1 (Part 2)

1. **Don't forget……**
a) The integral is the limit of Riemann sums.
b) The integral itself becomes a formula that enables us to solve problems without having to calculate the crudest estimate from a finite sum.
c) Whenever you want to find the cumulative effect of a varying rate of change, integrate.

2. **Review:** Write out the formula for the trapezoidal rule.

3. **Example 1:** A pump connected to a generator operates at a varying rate, depending on how much power is being drawn from the generator to operate other machinery. The rate (gallons per minute) at which the pump operates is recorded at 5-minute intervals for one hour as shown. How many gallons were pumped during that hour?

Time	0	5	10	15	20	25	30	35	40	45	50	55	60
Rate gal/min	58	60	65	64	58	57	55	55	59	60	60	63	63

4. Read the section on <u>Work</u> on Pg 384 and the margin note on **Joules.**

5. To find work, integrate the constant force formula.

6. **Example 2:** It takes a force of 10 N to stretch a spring 2m beyond its natural length. How much work is done in stretching the spring 4m from its natural length?

Assignment: Pg 386-388 #'s 21-31 odds, 34, 36 + Calculaugh #59 (Show detailed work on a separate page).

Section 7-2: Areas in the Plane

Objectives:
  • **Use integration to calculate areas of regions in a plane.**

1. **Area Between Curves**

Theory: Riemann Sums

This lesson is about finding the area of a region that is bounded by curves, both above and below. We can find the area as an integral, but lets first "see" the process.
1. We partition the region into vertical strips of equal width, making rectangles with base parallel to the x-axis. What are the dimensions of each rectangle? _____
2. We approximate the area by summing all the areas of these rectangles.
3. As the width of each rectangle goes to 0, the limit of these sums becomes the definition of the area between curves.
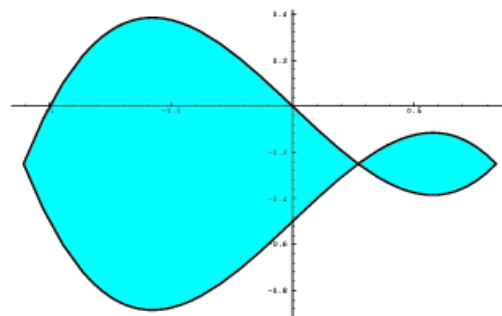
Definition: If f and g are continuous with $f(x) \geq g(x)$ throughout [a,b], then the area between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral of $[f - g]$ from a to b.

$$A = \int_a^b [f(x) - g(x)]dx$$

2. **Things to look out for:**

1. **Always** start with a sketch of the region involved. Label each part of the graph carefully.

2. You may be given an interval. Proceed with the definition to find the area.

3. When a region is closed by intersecting curves, the points of intersection give the limits. This means you have to find the points of intersection BEFORE you proceed with the definition. Use analytical/algebraic methods first and only if this is not possible, then use your calculator to find the intersections.

4. For step 3 above, a calculator can store the points of intersection and then you can evaluate using your calculator. However, my focus will always be algebraic.

5. A region can be bounded by more than one function. In this case, partition the region into sub-regions that correspond to the function changes and then proceed with multiple integrals.

6. Look for symmetry. The graph might be symmetric over the x-axis in which case you can evaluate one region, then multiply by 2.

3. Examples

1. Find the area of the region between $y = sec^2\ x$ and $y = sin\ x$ from $x = 0$ to $x = \dfrac{\pi}{4}$.

2. Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

3. Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Assignment: Pg 395 #'s 1-5, 7, 13, 15, 17, 28. **Note: Use your calc only on #7**

Section 7-2 (Contd): Horizontal Slices

1. Now for the fun part. We can slice a region vertically when one function is at the top of our section and a different function is at the bottom. But what if the same function is both the top and the bottom of the slice? This is called a **double-valued** funtion i.e. for every *x* there are two *y*-values. If we were to slice vertically we'd have a problem. For these situations you have to slice the region horizontally. Instead of integrating an equation *f(x)* with respect to *x*, we need to integrate an equation *f(y)* with respect to *y*. Sketches below to help you visualize this.

2. As a result, our area formula changes a little.

> If a region is bounded by f(y) on the right and g(y) on the left at all points of the interval [c, d], then the area of the region is given by:
> $$\int_{c}^{d}[f(y) - g(y)]dy$$

3. **Example 1**: Find the area of the region between the curve $x = y^2$ and the curve $x = y + 6$ from *y = 0* to *y = 3.*

Example 2: Find the area enclosed by the curves $y = x - 1$ and $y^2 = 2x + 6$.

Example 3: Use a calculator to find the area of the region enclosed by the graphs of $y = x^3$ and $x = y^2 - 2$.

Assignment: Worksheet Practice Problem Set

Section 7-3: Volumes of 3-d Figures using Integrals

Objectives:
- **Use integration to calculate volumes of solids.**
- **Use integration to calculate surface areas of solids of revolution.**

Nice pictures and examples at:

http://tutorial.math.lamar.edu/AllBrowsers/2413/VolumeWithRings.asp

This link is in Moodle under the VIDEO section.

1. **The Integral as an Infinite Sum**

- Key concept – The integral can be used to find the sum of an infinite number of smaller parts.
- Just like the area under a continuous curve can be approximated by a series of narrow rectangles, the volume of a solid of revolution can be approximated by a series of thin circular discs. (Volume = Area of Base x Height)
- We can improve our approximation by using a larger number of circular discs. Let us now establish a formula based on this idea:
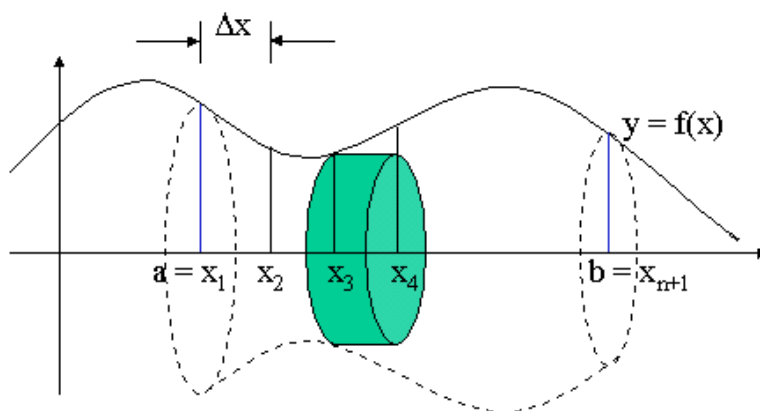
- Suppose we want to evaluate the volume of solid of revolution formed by the graph of $y = f(x)$ from $x = a$ to $x = b$. First of all, we divide the whole interval into n sub-intervals, all of which have a width of Δx. Then the i-th circular disc will have a radius of $f(x_i)$ and a thickness of Δx (the diagram above shows the third one).

$\therefore$ Volume of the i-th circular disc $= \pi[f(x_i)]^2 \Delta x$

$\therefore$ Sum of the volumes of all the discs $=$

$$\sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x$$

This approximate value will tend to the volume of solid of revolution if we let n tend to infinity.

$\therefore$ Volume of solid of revolution

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x$$

$$= \int_a^b \pi[f(x)]^2 dx$$

2. **Volume of a Solid – Definition**

- The **volume of a solid** with cross section area A(x) from x = a to x = b is: $\int_{a}^{b} A(x)dx$

3. **Example 1:** (*Rotate a region around the x-axis*) The region under the graph of $y=x^{1/3}$ from *x = 0* to *x = 8* is rotated about the *x-axis* to form a solid. Write an integral equal to the volume of the solid and evaluate it to find the volume.

Solution:
1. Draw a picture of the region.

2. Slice the region into strips, perpendicular to the axis of rotation.

3. Pick a sample point (x, y) that is on the graph and draw what a solid looks like as the region rotates around the axis.

4. The rotating strip creates a **disk**.

5. Find the volume, dV, for this disk.

6. The solid itself can be thought of as a stack of these disks. The exact volume is the limit of this Riemann sum, i.e. the definite integral.

7. Use calculus to integrate.

Example 2: (*Rotate a region around the y-axis*). The region in quadrant 1 bounded by the parabola $y = 4 - x^2$ is rotated about the *y-axis* to form a solid paraboloid. Find the volume of the paraboloid if *x* and *y* are in inches.

Assignment: Pg 406 #'s 2-6

Section 7-3 (Day 2) More Volume Methods

1. Review the technique we used yesterday to find volumes:

Volume of a solid by plane slicing

1. Cut the solid into flat slices, formed by strips in a rotated region or by planes
   passed through the solid. Get disks, washers, or slabs whose volumes can be
   found in terms of the solid's cross section at sample point(s) (x, y).
2. Do geometry to get dV in terms of the sample point(s).
3. Do algebra to get dV in terms of one variable.
4. Do calculus to add up all the dV's and take the limit (that is, integrate).
5. Check your answer to make sure it's reasonable.

Solving Solids of Revolution Using Washers:

Example 1: (*Rotate a region bounded by two curves*) Let R be the region that is bounded by the graphs of $y_1 = 6e^{-0.2x}$ and $y_2 = \sqrt{x}$, and by the vertical lines *x = 1* and *x = 4*. Find the volume of the solid generated when R is rotated about the *x-axis*.

Solution:
1. Sketch region *R*.

2. Slice it into strips,
    perpendicular to the axis of rotation.

3. Show a representative strip with two
    sample points, one on each graph.

4. As *R* rotates, the strip will trace out a
    disk with a hole in the middle, a **washer**.

5. The volume dV of the washer will be the
    volume of the outer disk minus the
    volume of the inner disk.

Read now the caution printed in your textbook on Pg 401. No excuses for making this mistake.

Example 2: The region in the first quadrant enclosed by the *y-axis* and the graphs of *y = cos x* and *y = sin x* is revolved about the *x-axis* to from a solid. Find its volume.

Solving Solids of Revolution using Cylinders

1. There is another way to find volumes of solids of rotation that can be useful when the axis of revolution is perpendicular to the axis containing the natural interval of integration. Instead of summing volumes of thin slices, we sum volumes of thin cylindrical shells that grow outward from the axis of revolution like tree rings.

Cylindrical Shells

A. **Steps:**

1. Sketch the region and draw a line segment across it parallel to the axis of revolution. (This line is the height of your cylinder)

2. Label its length (height of cylinder) and distance from axis of revolution (radius). The width of the segment is the shell thickness *dy* or *dx*

3. A(x) = Circumference ❌ r) x Height.
     V(x) = circumference(height)(width)

4. Identify the limits of integration according to the variable of integration and integrate for volume.
     Note: If we revolve about a line of form *x = *, we will integrate with respect to *x.* If we revolve about a line of form *y = *, we will integrate with respect to *y*.

B. **Another look at cylindrical shells**

1. Given is a region in quadrant 1 under the graph of
   $y = 4x - x^2$ from *x = 0* to *x = 3*. Suppose that this
   region is to be rotated about the *y-axis* to form a solid.

2. If you slice the region into strips parallel to the *y-axis*,
   the length of the strip is always equal to (*curve – line*).

3. As shown in the sketches, these parallel strips will
   generate cylindrical shells as the region rotates.

4. The shells are like tin cans without ends. Since a
   shell is thin, its volume dV can be found by cutting
   down its side and rolling it out flat. The rectangular
   solid will have the approximate dimensions shown.

5. Consequently, the volume of the shell is given by the
   property: dV = (circumference)(altitude)(thickness).

6. The volume of the solid will be equal to the sum of the
   volumes of the shells. The exact volume will be the
   limit of this sum i.e. the definite integral.

2. **Example 1:** The region under the graph of $y = 4x - x^2$ from *x = 0* to *x = 3* is rotated about the
   y-axis to from a solid. Find the volume of the solid by slicing into cylindrical shells. Use the FTC to
   obtain the exact answer.

3. **Example 2:** The region bounded by the curves $y = 4 - x^2$, $y = x$, and $x = 0$ is revolved about the *y-axis* to from a solid. Use cylindrical shells to find the volume of the solid.

Example 3: Let R be the region bounded by $y = x^{1/2}$, above by $y = 2$ and on the left by $y = x$. Use the shell method to find the volume of the solid rotated about the line $y = -1$. Assume x and y are in *ft*.

Assignment: Pg 406 #'s 7-9, 12-20 + Pg 407 #'s 22-26, 28, 29 (Do these as separate assignments)

 Section 7-3 (Day 4) Volumes Using Other Cross Sections

1. We can use the method of cross-section slicing to find volumes of a wide variety of unusually shaped solids, so long as the cross sections have areas that we can describe with some formula.

2. **Example 1:** A mathematician has a paperweight made so that its base is the shape of the region between the x-axis and one arch of the curve $y = 2\sin(x)$. Each cross section cut perpendicular to the x-axis is a semicircle whose diameter runs from the x-axis to the curve. (Think of the cross section as a semicircular fin sticking up out of the plane). Find the volume of the paperweight.

3. **Cavalieri's Volume Theorem:** Remember this from geometry?

   This theorem says that solids with equal altitudes and identical cross section areas at each height have the same volume.

4. Cylindrical shells can be used to find volumes when these conditions are encountered:
   - The rotation is not around the y-axis.
   - The axis of rotation is not a bound of integration.
   - Both ends of the shell's altitude are variable.

Assignment: Pg 407 #'s 27, 30, 33, 35, 37, 45, 46

Section 7-4: Lengths of Curves

Objectives:

- **Use integration to calculate lengths of curves in a plane.**

1. In this section you will find the length of a curve, sometimes known as arc length.

The Process: We partition the curve into intervals so short that the pieces of curve are nearly straight. Using the Pythagorean theorem we can calculate the length of this line that closely approximates the curve. The sum of several such segments approximates the length of the curve. All we need now is to find the limit of this sum as the norms of the partitions goes to zero. However, as you will see in our example below, these sums are not Riemann sums. With a little algebra manipulation we should be able to convert our formula.

Example for formula:

We are almost ready for a formal definition. A couple of properties to consider:
a) The function should be differentiable for the interval defined.
b) The function should have a continuous first derivative. We call this property **smoothness**.
c) The graph of the function should be a smooth curve.
2. **Definition :** If a smooth curve begins at (a, c) and ends at (b, d), then the length (arc length) of the curve is:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \text{ if } y \text{ is a smooth function of } x \text{ on } [a,b].$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \text{ if } x \text{ is a smooth function of } y \text{ on } [c,d]$$

3. **Example 1:** Find the exact length of the curve $y = \frac{4\sqrt{2}}{3} x^{3/2} - 1$, for $0 \le x \le 1$.

4. **Vertical Tangents, Corners and Cusps:**
a) If the curve has a vertical tangent (where the derivative becomes infinite) over the interval, change the variable of integration.
b) If the curve has a corner or a cusp, split the integral and add the two curve lengths.

5. **Example 2:** Find the length of the curve $y = x^{1/3}$ between (-8, -2) and (8, 2).

Assignment: Pg 416 #'s 1, 3, 5, 7, 11, 13, 17, 23, 24

Section 7-5: Applications from Science and Statistics – (skip Statistics)

Objectives:
*   **Adapt knowledge of integral calculus to model problems involving rates of change in a variety of applications.**

1. Recall from 7-1 that work is defined as force (in the direction of motion) times displacement.

$$W = \int_a^b F(x)dx$$

So in general, develop a function from the information given, and then integrate it.

2. You will also need **Hooke's Law** for springs which says that the force it takes to stretch or compress a spring x units from its natural length is a constant times x. In symbols:

$$F = kx \text{ , where k is measured in force units per unit length.}$$

Example 1: A leaky bucket weighs 22 newtons (N) empty. It is lifted from the ground at a constant rate to a point 20 m above the ground by a rope weighing 0.4 N/m. The bucket starts with 70 N of water, but it leaks at a constant rate and just finishes draining as the bucket reaches the top. Find the amount of work done:
a) lifting the bucket alone.
b) lifting the water alone.
c) lifting the rope alone.
d) lifting the bucket, water and rope together.

Statistics:

3. **Normal Probabilities:** Mathematicians represent probabilities as area, and that is where definite integrals enter the picture. Probabilities of events are integrals of an appropriate *pdf,* which is a *probability density function*.

4. The normal probability density function (Gaussian curve) for a population with mean μ and standard deviation σ is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

5. The mean represents the average value of x. The standard deviation measures the "scatter" around the mean. For a normal curve, the mean and standard deviation tell you where most of the probability lie. The rule for normal distributions is given below:

The 68-95-99.7 Rule:

Given a normal curve,
- 68% of the area will lie within 1 s.d. of the mean
- 95% of the area will lie within 2 s.d. of the mean
- 99.7% of the area will lie within 3 s.d. of the mean.

Assignment: Pg 425 #'s 1, 3, 5, 7, 9, 10

Chapter 7 Review

The test will take place in two parts on the 80-minute day. The first part will be the non-calculator test, followed by the calculator test. The following topics will be covered on the test.

You will be responsible for your time – i.e. I will **NOT** tell you when to hand in one test and pick up the other. You may only have **one** test on your desk at any given time. Once the calculator test has been turned in, you may **NOT** take it back to make changes on that test. Both parts of the test must be completed within the one class period.

- Integral as net change
- Particle motion
- Work: 7-1 and 7-5 sections
- Area between curves
- Volume – all methods including various other cross sections
- Lengths of curves
- Exponential growth and decay – Chapter 6
- Trapezoidal Reimann sum – Redo Pg 387 # 27

Review assignment: Pg 430 #'s 1, 4, 7, 10, 17, 20, 21, 24, 28, 29, 34, 44.