CHAPTER 6: DIFFERENTIAL EQUATIONS AND MATHEMATICAL MODELING

Section 6-1: Slope Fields and Euler's Method

Objectives:

- **Construct antiderivatives using the FTC.**
- **Solve initial value problems.**
- **Construct slope fields and interpret slope fields as visualizations of different equations.**

1. **Definition:** An equation involving a derivative is called a **differential equation**. The **order** of a differential equation is the order of the highest derivative involved in the equation.

Example 1: Find the general solution to the exact differential equation.

$$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x^2}$$

2. **Initial Value Problems**
   - We often want to find the original function of x given its **derivative** and given a point (x,y).
   - When we use the point-slope formula, we do exactly this for a linear function.

   - **<u>A very important note:</u>** If $f(x)$ is continuous, the initial condition pins down the curve in the entire domain, **BUT** if $f(x)$ is discontinuous, the initial condition only pins down that piece of the graph that passes through that point. In this case the domain MUST be specified.

Example 2: Find the solution to the differential equation $f'(x) = 2e^x - \cos x$ for which *f(0) = 3*.

Example 3: Find the particular solution to the equation $\frac{dy}{dx} = e^x - 6x^2$ whose graph passes through the point (1, 0).

3. **Solving an initial value problem when we are unable to find an antiderivative…**
(Use the Fundamental Theorem - your answer will contain a definite integral)

Example 4: $\frac{dy}{dx} = \sqrt{2 + \cos x}$ and y = -3 when x = 0

Example 2: Find the solution to the differential equation $f'(x) = e^{-x^2}$ for which $f(7) = 3$.

4. **Graphing a General Solution**

Graph the family of functions that solve the differential equation $\frac{dy}{dx} = \cos(x)$.

Calculator keys: **Sketch**

5. **Introduction to Slope Fields**

1. Do Exploration 1 on page 323.
2. Read through page 323 and work through example 6 and through to page 324.
3. Does your calculator have the capability of drawing slope fields?

Assignment: Pg 327 #'s 1-23 odds

Section 6-1 Contd: Slope Fields

A **slope field** will let you solve complicated differential equations graphically, whereas **antiderivatives** let you solve them algebraically. **Euler's method** is a way to solve differential equations numerically. A **slope field** is another way to show a "family of antiderivatives".

Use the following examples of differential equations to draw slope fields. For each, make a conjecture/conclusion.

1. $\frac{dy}{dx} = 2$

Conclusion:

2. $\frac{dy}{dx} = x + 1$

Conclusion:

3. $\frac{dy}{dx} = 2y$

Conclusion:

4. $\frac{dy}{dx} = x + y$

Conclusion:

Assignment: Slope fields worksheet

Section 6-2: Antidifferentiation by Substitution

Objectives:

- **Compute definite and indefinite integrals by the method of substitution.**

1. **Indefinite Integrals**

A. $\int f(x)dx$ denotes the family of all anti-derivatives of f(x) and is called the ________________

____

x → Variable of Integration

$\int$ → Integral Sign

f → Function called the Integrand of the Integral

B. **Definite Integral** **vs.** **Indefinite Integral**

C. **Examples:** $\int \frac{x^5+7}{x^2}dx$ $\int \sec x(\sec x+\tan x)dx$

2. **U – Substitution (Reverse Chain Rule)**

A. **How?** $\int f(g(x))g'(x)dx =$

Step 1: Substitute *u* = *g(x)* and *du* = dx so that you have $\int f(u)du$.
Step 2: Integrate with respect to *u.*
Step 3: Replace *u* by *g(x)* in the result.

B. **Examples:**

$\int (5x+7)^{14}dx$; Let *u* = *5x* + *7*

$\int x(4x^2-7)^{10}dx$; Let *u* = *4x*^{2} – *7*

$\int x\cos(3x^2+1)dx$; Let $u = 3x^2 + 1$

$\int \sin(\sin x)\cos x\, dx$; Let $u = \sin x$

$\int \sin 2x \cos 2x\, dx$

$\int \frac{3x\ dx}{\sqrt[3]{10-x^2}}$

$\int \cos^2 x\ dx$

(using Trig. Identity: $\cos^2 x = \frac{1+\cos 2x}{2}$)

$\int_0^1 \frac{x}{x^2-4}dx$

Assignment: Pg 337 #'s 1-6,17-24

Trig Identities and More Examples for Section 6-2:

Fundamental Identities

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Addition and Subtraction Formulas

$\sin(x+y) = \sin x\cos y + \cos x\sin y$

$\sin(x-y) = \sin x\cos y - \cos x\sin y$

$\cos(x+y) = \cos x\cos y - \sin x\sin y$

$\cos(x-y) = \cos x\cos y + \sin x\sin y$

$\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$

$\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x\tan y}$

Double-Angle Formulas

$\sin 2x = 2\sin x\cos x$

$\cos 2x = 1 - 2\sin^2 x$

$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

Half-Angle Formulas

$\sin^2 x = \dfrac{1 - \cos 2x}{2}$ $\qquad$ $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

A Note on Absolute Value: Since the indefinite integral does not specify a domain, you should always use the absolute value when finding $\int \frac{1}{u}du$. The function $\ln u + C$ is only defined on positive *u*-intervals, while the function $\ln|u| + C$ is defined on both the positive and negative intervals in the domain of $1/u$.

Example 1: $\int \cot(7x)dx$

Example 2: $\int \cot^2(3x)dx$

Assignment: Pg 338 #'s 25-43 odds, 49, 55, 61, 65

Section 6-4: Exponential Growth and Decay

Objectives:

- **Solve problems involving exponential growth and decay in a variety of applications.**

Part 1: Separable Differential Equations

1. **Definition:** A differential equation of the form $\frac{dy}{dx} = f(y)g(x)$ is called separable. We separate the variables by writing it in the form $\frac{1}{f(y)}dy = g(x)dx$. The solution is found by anti-differentiating each side with respect to its thusly isolated variable.

Example 1: Solve for y if $dy/dx = (xy)^2$ and $y = 1$ when $x = 1$.

Example 2: Solve for *y* if $\frac{dy}{dx} = \frac{-x}{y}$ and $y = 3$ when $x = 4$.

Assignment: Pg 357 #'s 1-9 odds + Calculaugh # 52 (Show work on a different paper)

Part 2: Exponential Growth and Decay

1. **Law of Exponential Change**:
   - Is used for all problems when growth/decay happens at a proportional rate i.e. the rate of change is proportional to the amount present.
   - The differential equation that describes this is $dy/dt = ky$ where k is the growth (if positive) or decay (if negative) constant.
   - Types of problems include compounding interest, radioactive decay, Newton's Law of Cooling etc.

2. **Definition of Half-Life:** The half-life of a radioactive substance with rate constant $k (k > 0)$ is half-life = $\frac{\ln 2}{k}$.

3. Show that the differential equation $dy/dt = ky$ results in the familiar formula $y = Ae^{kt}$.

4. **Newton's Law of Cooling:** is $T - T_S = (T_0 - T_s)e^{-kt}$.

Example: A hard-boiled egg at $98^\circ C$ is put in a pan under running $18^\circ C$ water to cool. After 5 mins, the egg's temperature is found to be $38^\circ C$. How much longer will it take the egg to reach $20^\circ C$?

NOTE: You should memorize that carbon-14 has a half-life of 5700 years.

Assignment: Pg 357 #'s 22-38 evens. Note: 36 a & b only.

CHAPTER 6 TEST REVIEW

The test will take two days. The first day will be no calculator. The following topics will be covered on the test.

- Integration: Definite vs Indefinite
- Memorization of rules
- U-Substitution
- Initial Value Problems
- Separable differential equations
- Applied problems
- Slope fields

Assignment: Pg 373 #'s 3, 6, 8, 12, 15, 17, 18, 26, 27, 31, 32, 35, 37, 49, 52, 63. (In #'s 31, 32, solve for y)